# computation with qudits (entangled or otherwise)

**scalable quantum research lab**

group meeting

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**Definition 1**

A givens rotation is represented by a matrix of the form

$$\mathbf{G}(i,j,\theta) = \begin{bmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & c & \dots & -s & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & s & \dots & c & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & s & \dots & c & \dots & 0 \end{bmatrix}$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$. Here $c, s \in \mathbb{R}$ but generalization to $c, s \in \mathbb{C}$ can be made.

**Theorem 1**

*$\mathbf{G}(i, j, \theta)$ is orthonormal and generalizations with complex valued entries are unitary*

**Theorem 2**

*Any matrix $A \in \mathbb{R}^{d \times d}$ can be written as a product of an orthonormal matrix and an upper triangular matrix, i.e.*

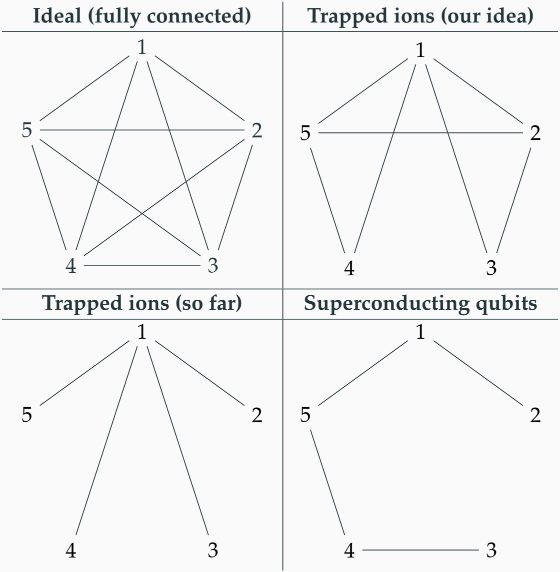
$$\mathbf{A} = \mathbf{Q}\mathbf{R}$$

*When the starting matrix is unitary, there is an added consequence that $\mathbf{R}$ is diagonal.*

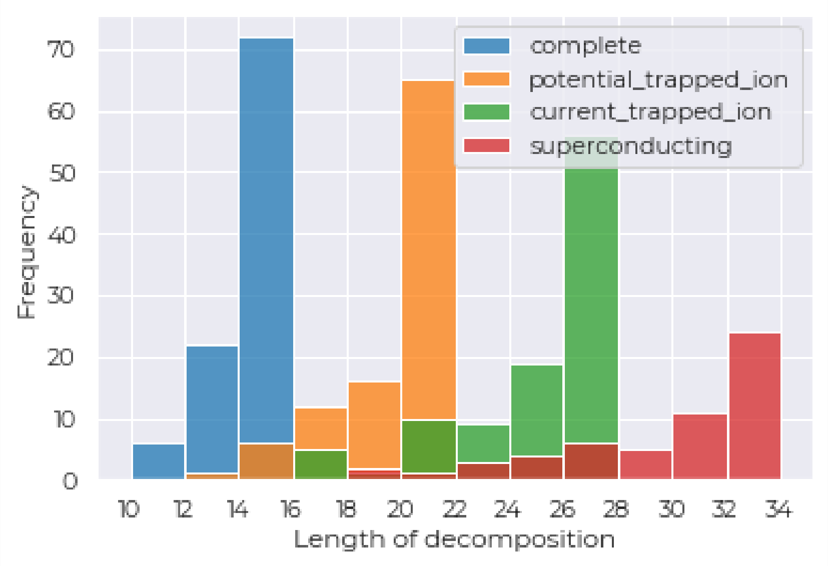
**Proof.**

Not an actual proof but the idea is simply for a given $i, j$ to select an appropriate givens matrix and on multiplying by the left with $\mathbf{A}$; can zero out an entry along the main diagonal. □

# Connectivity graphs



# Decomposition Lengths



# Fidelity Analysis (a sketch)

Each givens rotation can be written as $\exp(i\theta\mathbf{H})$ for some hermitian $\mathbf{H}$ and some angle $\theta$. We've dealt with decomposition length so we now focus on primary fidelity. We now assume that physically we can realize each givens rotation with some rotation angle error namely $\exp(i(\theta+\delta)\mathbf{H})$. More concretely let $k$ be the decomposition length of the unitary that we'd like to implement, thereby giving us

$$\mathbf{U}_{\text{ideal}} = \prod_{i=1}^{k} \exp(i\theta_i \mathbf{H_i}) = \prod_{i=1}^{k} \mathbf{U}_i$$

and

$$\mathbf{U}_{\text{ion}} = \prod_{i=1}^{k} \exp(i(\theta_i+\delta_i)\mathbf{H}_i) = \prod_{i=1}^{k} \exp(i\theta_i\mathbf{H}_i)\exp(i\delta_i\mathbf{H}_i) = \prod_{i=1}^{k} \mathbf{U}_i\mathcal{E}_i$$

# fidelity analysis (a sketch)

Assume some arbitrary starting state $|\psi\rangle$, then

$$
\begin{aligned}
F &= \left| \left\langle \psi \middle| \mathbf{U}_{\text{ideal}}^{\dagger} \mathbf{U}_{\text{ion}} \left|\psi\right\rangle \right\rangle \right| = \left| \left\langle \psi \middle| \left( \prod_{i=1}^{k} \mathbf{U}_i \right)^{\dagger} \left( \prod_{i=1}^{k} \mathbf{U}_i \mathcal{E}_i \right) \left|\psi\right\rangle \right\rangle \right| \\
&\geq \left| \left\langle \psi \middle| \prod_{i=1}^{k} \mathcal{E}_i \left|\psi\right\rangle \right\rangle \right| \geq \left| \langle \psi | \exp\left( i \cdot k\alpha \cdot \mathbf{H}_i \right) |\psi\rangle\rangle \right| \\
&\approx \left( 1 - \frac{k^2 \alpha^2}{2} \right)^2
\end{aligned}
$$

Matches intuition somewhat (calculation not as important as result where fidelity depends on the decomposition length and rotation angle error)

# Characterizing multi-level qudit entanglement

# A motivating example

Consider the two **ququart** "maximally entangled" state

$$|\psi_4\rangle = \frac{1}{2}\left(|00\rangle + |11\rangle + |22\rangle + |33\rangle\right)$$

However, given two copies of the **two-qubit** Bell state $|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$, the two states $|\psi_2\rangle$ and $|\psi_4\rangle$ are equivalent up to relabelling.

$$\begin{aligned}|\psi_2\rangle|\psi_2\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle_A + |11\rangle_A\right)\frac{1}{\sqrt{2}}\left(|00\rangle_B + |11\rangle_B\right)\\ &= \frac{1}{2}\left(|00\rangle_A|00\rangle_B + |00\rangle_A|11\rangle_B + |11\rangle_A|00\rangle_B + |11\rangle_A|11\rangle_B\right)\end{aligned}$$

Therefore under the identification scheme

$$|00\rangle_{A,B} \mapsto |0\rangle_C, |01\rangle_{A,B} \mapsto |1\rangle_C, |10\rangle_{A,B} \mapsto |2\rangle_C, |11\rangle_{A,B} \mapsto |3\rangle_C$$

the above state "becomes"

$$|\psi_2\rangle|\psi_2\rangle = \frac{1}{2}\left(|00\rangle + |11\rangle + |22\rangle + |33\rangle\right)$$

# Some notes about what we just did

1. This is good! (don't quote me on this) since one can easily prepare $|\psi_2\rangle^{\otimes n}$ which in this view is "equivalent" to the maximally entangled state in dimension $2^n \times 2^n$.
2. We left some key details out. Qualitatively larger Hilbert spaces are more "rich" so to speak. To have "high-dimensional entanglement" (whatever this means) we must be able to perform arbitrary local measurements (joint or otherwise).
3. Finally being able to implement arbitrary local transformations on the four dimensional gate would also be nice.
4. Many different ways to try and quantify high-dimensional entanglement (entanglement witnesses; Schmidt rank of a system, etc.)

## The scenario

For demonstration and intuition; we work with four-level systems (**ququarts**). A general two-ququart entangled state can be written in decomposition

$$|\psi\rangle = s_0 |00\rangle_{A,B} + s_1 |11\rangle_{A,B} + s_2 |22\rangle_{A,B} + s_3 |33\rangle_{A,B} \text{ where } \sum_{i=0}^{3} s_i^2 = 1$$

**Question:** Can we replace each ququart with two qubits so that your total state can be considered a 4 qubit state as in the example we did above. The first idea we have is to do what we did in the example above simply; replace on Alice's side $|0\rangle \mapsto |00\rangle$, $|1\rangle \mapsto |01\rangle$, $|2\rangle \mapsto |10\rangle$ and $|3\rangle \mapsto |11\rangle$ and similarly for Bob. Then this replacement leaves us with

$$\begin{aligned} |\psi\rangle = s_0 |00\rangle_{A_1,B_1} |00\rangle_{A_2,B_2} + s_1 |00\rangle_{A_1,B_1} |11\rangle_{A_2,B_2} \\ + s_2 |11\rangle_{A_1,B_1} |00\rangle_{A_2,B_2} + s_3 |11\rangle_{A_1,B_1} |11\rangle_{A_2,B_2} \end{aligned}$$

# Approach and the result

Continuing what we had, we ask the conditions in which the state can be written down as

$$|\varphi\rangle = (\alpha_0 |00\rangle_{A_1,B_1} + \alpha_1 |11\rangle_{A_1,B_1}) \otimes (\beta_0 |00\rangle_{A_2,B_2} + \beta_1 |11\rangle_{A_2,B_2})$$

with a similar identification scheme we discussed above as in the motivating example. If $|\psi\rangle$ can be written in this form, then it is said to be decomposable; otherwise, it is called genuinely four-level entangled. In order to decide decomposability for a general $|\psi\rangle$, we can compute the maximum overlap between $|\psi\rangle$ and a general decomposable state $|\varphi\rangle$, i.e.

$$\begin{aligned}\max_{|\varphi\rangle} |\langle\psi|\varphi\rangle| &= \max_{\alpha_i,\beta_i} (s_0\alpha_0\beta_0 + s_1\alpha_0\beta_1 + s_2\alpha_1\beta_0 + s_3\alpha_1\beta_1) \\ &= \max_{\boldsymbol{\alpha},\boldsymbol{\beta}} \langle\boldsymbol{\beta}|S|\boldsymbol{\alpha}\rangle = \max \text{singval}(S) \\ \text{defining } S &= \begin{bmatrix} s_0 & s_1 \\ s_2 & s_3 \end{bmatrix} \text{ and } \boldsymbol{\alpha} = (\alpha_0, \alpha_1)^\top \boldsymbol{\beta} = (\beta_0, \beta_1)^\top\end{aligned}$$

# Some results and generalizations

**Theorem 3**

*The two ququart state $|\psi\rangle$ is decomposable if and only if* $\max singval(S) = 1$.

- A mixed state is decomposable iff $\varrho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ where $|\psi_i\rangle$ are decomposable and genuine four-level entangled otherwise.
- Convexity of the set of decomposable states allows one to construct "witnesses" for four-level entanglement (Hahn-Banach Separation Theorem).
- An entanglement witness $\mathcal{W}$ is such that $\mathrm{tr}(\varrho\mathcal{W}) \geq 0$ for all separable states and therefore $\mathrm{tr}(\varrho\mathcal{W}) < 0$ signals some sort of genuine "multi-partite entanglement".

**Example 1**

A class of witnesses are the projector-based witnesses namely $\mathcal{W} = \alpha\mathbf{1} - |\xi\rangle\langle\xi|$

# A final note on generalizations

1. In the setting of considering decomposition into 2-lower dimensional states; the results from the previous section still hold (the only difference is that the matrix $S$ increases in dimension)
2. Now singular values depend on the encoding which thereby determines on the Schmidt coefficients that occur in the matrix $S$.

**Definition 2**
An $N$-partite pure state $|\psi\rangle \in (\mathbb{C}^D)^{\otimes N}$ is fully-decomposable iff there exist $N$-partite states $\varphi, \varphi'$ with dimension $d, d'$ such that

$$|\psi\rangle = U_1 \otimes \cdots \otimes U_n(|\varphi\rangle \otimes |\varphi'\rangle)$$

# Qudit universality

# A small universal gate set

Let $\mathbf{U}_d$ be a $d$-dimensional transformation mapping a general qudit state to $|d-1\rangle$, i.e.

$$\mathbf{U}_d(\alpha_0, \ldots, \alpha_{d-1}) : \sum_{i=0}^{d-1} \alpha_j |j\rangle \mapsto |d-1\rangle$$

We claim that

$$\mathbf{U}_d = \mathbf{X}_{d-1}(a_{d-1}, b_{d-1}) \ldots \mathbf{X}_1(a_1, b_1)$$

where $a_i = \alpha_i$ and $b_i = \sum_{j=0}^{i-1} \alpha_j^2$ and

$$\mathbf{X}_i(a_i, b_i) = \mathbf{G}\left(i-1, d-j-1, \arccos\left(\frac{a_i}{\sqrt{|a_i|^2 + |b_i|^2}}\right)\right)$$

Additionally we define the $d$-dimensional phase gate $\mathbf{Z}_d$ such that

$$\mathbf{Z}_d(\theta) = \sum_{j=0}^{d-1} e^{i(1-\operatorname{sgn}(d-j-1))\theta} |j\rangle\langle j| \text{ alters the phase of } |d-1\rangle \text{ by } \theta$$

# Universal gate sets

Finally we define $C_2[\mathbf{R}_d]$ where $\mathbf{R}_d$ is either $\mathbf{X}_d$ or $\mathbf{Z}_d$ as

$$C_2[\mathbf{R}_d] = \begin{pmatrix} \mathbf{I}_{d^2-d} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_d \end{pmatrix}$$

**Theorem 4**

*The following gate set*

$$\Gamma = \{\mathbf{X}_d, \mathbf{Z}_d, C_2[\mathbf{R}_d]\}$$

*is universal for general quantum computation based on the circuit model.*

Note here critically that universality in this setting implies addressing $n$-qudit unitary operations in $\mathsf{SU}(d^n)$ as opposed to single-qudit operations in $\mathsf{SU}(d)$. Let us denote the computational basis of $n$-qudit space $\mathbb{C}^{d^n}$ as

$$|k\rangle = |k_1 \ldots k_n\rangle$$

where $k_1 \ldots k_n$ is the base-$d$ representation of $k$ and

# A proof sketch

**Step 1: Eigen-decomposition of U**

From the spectral theorem we can write $\mathbf{U}$ as

$$\mathbf{U} = \sum_{j=1}^{N} e^{i\lambda_j} |E_j\rangle\langle E_j| = \prod_{j=1}^{N} \sum_{s=1}^{N} e^{i(1-\text{sgn}(j-s))\lambda_s} |E_s\rangle\langle E_s| = \prod_{j=1}^{N} \Upsilon_j$$

where $\Upsilon_j$ adds a phase of $\lambda_j$ to the corresponding eigenstate and leaves all the other eigenstates unchanged. Crucially,

$$\Upsilon_j = \mathbf{U}_{j,N}^{-1} \mathbf{Z}_{j,N} \mathbf{U}_{j,N}$$

where $\mathbf{U}_{j,N}, \mathbf{Z}_{j,N}$ are the $N$-dimensional analogues of $\mathbf{U}_d, \mathbf{Z}_d$, i.e.

$$\mathbf{U}_{j,N}(\alpha_0, \ldots, \alpha_{n-1}) : |E_j\rangle \mapsto |N-1\rangle$$

and

$$\mathbf{Z}_{j,N} = \sum_{s=0}^{N-1} e^{i(1-|\text{sgn}(s-N+1)|)\theta} |s\rangle\langle s| \text{ alters the phase of } |N-1\rangle \text{ by } \lambda_j$$

# A proof sketch

**Step 2: Controlled decomposition of $\mathbf{U}_{j,N}$ and $\mathbf{Z}_{j,N}$**
Similar to the decomposition of $\mathbf{U}_d$, $\mathbf{U}_{j,N}$ can be decomposed in terms of primitve gates $X_{j,k}(x,y)$. Finally $\mathbf{Z}_{j,N} = C_n[\mathbf{Z}_d(\lambda_j)]$ where

$$C_n[\mathbf{R}_d] = \begin{pmatrix} \mathbf{I}_{d^k-d} & \\ & \mathbf{R}_d \end{pmatrix}$$

$\mathbf{R}_d$ acts on the last $d$ substates $\{|d-1\rangle^{\otimes n-1}|i\rangle\}_{i\in[n-1]}$. This controlled qudit operation transforms the last qudit by applying $\mathbf{R}_d$ conditional on the first $k$ qudits being in $|d-1,\dots,d-1\rangle$

**Step 3: Primitive decomposition of $C_m[\mathbf{R}_d]$**
Can build a circuit using $\lceil \frac{m-2}{d-2} \rceil$ auxiliary qudits, multiple applications of $C_2$ and a final application of $\mathbf{R}_d$ in order to replicate the behavior of $C_m[\mathbf{R}_d]$

# Future directions

- There seem to be subtle differences between decomposability and separability; stuff I think that's useful.
- Would like to understand the concept of Schmidt rank better and its connection to the SVD (comes out of the singular value decomposition, the coefficients that come out of a bipartite pure state are treated as a "matrix".)
- It turns out that for bipartite systems of dimension $2 \otimes 2$ and $2 \otimes 3$; there is a necessary and sufficient condition for separability (PPT condition).
- Would be nice if we could do universal qudit computation without any auxiliary qudits; once we have bearings on this, can start to think about the length of gate decomposition like for the single qudit case.

# References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