

A matching based algorithm for Quantum Max-Cut

September 5, 2026

(Classical) Max-Cut

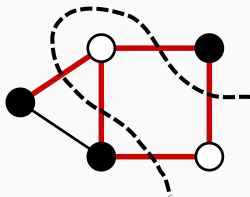


Figure 1: max cut is highlighted in red; size 5

- Intuitively, terms in the sum only have non-zero contribution when $z_i \neq z_j$, so the z_i 's of the same sign identify the partition.

(Classical) Max-Cut

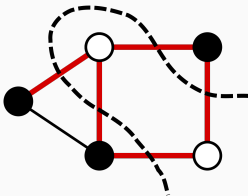


Figure 1: max cut is highlighted in red; size 5

- **Goal:** Given $G = (V, E)$, find some $S \subseteq V$ such that the partition maximizes

$$\sum_{\substack{(i,j) \in E \\ z_i \in \{\pm 1\}}} \left\{ \frac{1 - z_i z_j}{2} \right\}$$

- Intuitively, terms in the sum only have non-zero contribution when $z_i \neq z_j$, so the z_i 's of the same sign identify the partition.

(Classical) Max-Cut

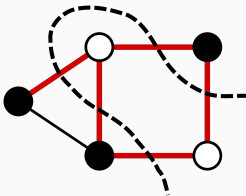


Figure 1: max cut is highlighted in red; size 5

- **Goal:** Given $G = (V, E)$, find some $S \subseteq V$ such that the partition maximizes

$$\sum_{\substack{(i,j) \in E \\ z_i \in \{\pm 1\}}} \left\{ \frac{1 - z_i z_j}{2} \right\}$$

- Intuitively, terms in the sum only have non-zero contribution when $z_i \neq z_j$, so the z_i 's of the same sign identify the partition.
- Known to be **NP-hard** to obtain an exact solution :(

Local Hamiltonians

- Given some $G = (V, E)$ on n vertices, we can think of the vertices as qubits where the edges of this "interaction graph" describe our local hamiltonian terms, i.e.

$$\mathbf{H} = \sum_{(u,v) \in E} \mathbf{H}_{u,v} \otimes \mathbb{I}_{V \setminus \{u,v\}}$$

Goal: Find a quantum state $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$ that returns the maximum/minimum energy or eigenvalue of $\mathbf{H}$, i.e.

$$\text{energy} = \langle \psi | \mathbf{H} | \psi \rangle$$

Local Hamiltonians

- Given some $G = (V, E)$ on n vertices, we can think of the vertices as qubits where the edges of this "interaction graph" describe our local hamiltonian terms, i.e.

$$\mathbf{H} = \sum_{(u,v) \in E} \mathbf{H}_{u,v} \otimes \mathbb{I}_{V \setminus \{u,v\}}$$

Goal: Find a quantum state $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$ that returns the maximum/minimum energy or eigenvalue of $\mathbf{H}$, i.e.

$$\text{energy} = \langle \psi | \mathbf{H} | \psi \rangle$$

- One of the quantum analogues of classically well-understood constraint satisfaction problems.

(Quantum) Max-Cut

- **Input:** 2-local Hamiltonian,

$$H = \sum_{(i,j) \in E} \frac{\mathbb{I} - X_i X_j - Y_i Y_j - Z_i Z_j}{4}$$

- **Goal:** Return $\lambda_{\max}(H)$

(Quantum) Max-Cut

- **Input:** 2-local Hamiltonian,

$$H = \sum_{(i,j) \in E} \frac{\mathbb{I} - X_i X_j - Y_i Y_j - Z_i Z_j}{4}$$

- **Goal:** Return $\lambda_{\max}(H)$

Why is this a generalization? Recall that in the classical problem, a constraint had maximal energy when two corresponding entries disagreed. So, what's our quantum notion of "disagreement"?

(Quantum) Max-Cut

- **Input:** 2-local Hamiltonian,

$$H = \sum_{(i,j) \in E} \frac{\mathbb{I} - X_i X_j - Y_i Y_j - Z_i Z_j}{4}$$

- **Goal:** Return $\lambda_{\max}(H)$

Why is this a generalization? Recall that in the classical problem, a constraint had maximal energy when two corresponding entries disagreed. So, what's our quantum notion of "disagreement"? From the lovely algebraic identity

$$\text{SWAP}_{i,j} = \frac{\mathbb{I} + X_i X_j + Y_i Y_j + Z_i Z_j}{2} \implies H_{i,j} = \frac{\mathbb{I} - \text{SWAP}_{i,j}}{2}$$

(Quantum) Max-Cut

- **Input:** 2-local Hamiltonian,

$$H = \sum_{(i,j) \in E} \frac{\mathbb{I} - X_i X_j - Y_i Y_j - Z_i Z_j}{4}$$

- **Goal:** Return $\lambda_{\max}(H)$

Why is this a generalization? Recall that in the classical problem, a constraint had maximal energy when two corresponding entries disagreed. So, what's our quantum notion of "disagreement"? From the lovely algebraic identity

$$\text{SWAP}_{i,j} = \frac{\mathbb{I} + X_i X_j + Y_i Y_j + Z_i Z_j}{2} \implies H_{i,j} = \frac{\mathbb{I} - \text{SWAP}_{i,j}}{2}$$

Exactly the anti-symmetric subspace! So energy is maximal when $\text{SWAP}_{i,j} |\psi\rangle = -|\psi\rangle$

Some more quantum intuition

$$H = \sum_{(i,j) \in E} \frac{\mathbb{I} - X_i X_j - Y_i Y_j - Z_i Z_j}{4}$$

Some more quantum intuition

$$H = \sum_{(i,j) \in E} \frac{\mathbb{I} - X_i X_j - Y_i Y_j - Z_i Z_j}{4}$$

- The $X_i X_j$ terms makes a measurement in the X basis.
 - -1 if both are the same eigenstate, i.e. $|++\rangle$ or $|--\rangle$.
 - $+1$ if both are in differing eigenstates $| - + \rangle, | + - \rangle$.

Some more quantum intuition

$$H = \sum_{(i,j) \in E} \frac{\mathbb{I} - X_i X_j - Y_i Y_j - Z_i Z_j}{4}$$

- **The** $X_i X_j$ terms makes a measurement in the **X** basis.
 - -1 if both are the same eigenstate, i.e. $|++\rangle$ or $|--\rangle$.
 - $+1$ if both are in differing eigenstates $| - + \rangle, | + - \rangle$.
- In order to maximize, **The** $Y_i Y_j$ term should be different in the **Y** basis and similarly for the $Z_i Z_j$ term.
- Like classical max-cut but not just in the **Z** basis.

Approximation algorithms for QMC

- Product state algorithm that uses an SDP approach to obtain a 0.498 approximation by Gharibian and Parekh [2].
- Tensor product of one and two qubit states that achieves a 0.531 approximation ratio by Anshu, Gosset and Morenz [1].
- Parekh and Thompson provided a 0.533 approximation algorithm in 2020 [6]
- The first algorithm to incorporate entanglement was due to King in 2022 achieving an approximation ratio of 0.582 for triangle-free graphs. [4]
- Finally Parekh and Lee provide a 0.595 matching-based approximation algorithm [5]

Like Parekh and Lee, we look to maximum matching within graphs and manage to improve their analysis to yield a 0.606-approximation algorithm for quantum max-cut, and if the graph is promised to be bipartite, we can guarantee a 0.75 approximation.

Crucially, the result for bipartite instances may be somewhat surprising to folks (at least it was to me) since classically these are the easiest instances of the problem.

Bipartite instances and EPR

So about that bipartite guarantee...?

Bipartite instances and EPR

So about that bipartite guarantee...? First a short detour,

$$\text{EPR} = \sum_{(i,j) \in E} \frac{\mathbb{I} + X_i X_j - Y_i Y_j + Z_i Z_j}{4} = \sum_{(i,j) \in E} |\Phi\rangle\langle\Phi|_{i,j}$$

Why bother with this family of Hamiltonians? Well theoretically interesting but also rotationally (unitarily) related to the original max-cut Hamiltonian.

Bipartite instances and EPR

So about that bipartite guarantee...? First a short detour,

$$\text{EPR} = \sum_{(i,j) \in E} \frac{\mathbb{I} + X_i X_j - Y_i Y_j + Z_i Z_j}{4} = \sum_{(i,j) \in E} |\Phi\rangle\langle\Phi|_{i,j}$$

Why bother with this family of Hamiltonians? Well theoretically interesting but also rotationally (unitarily) related to the original max-cut Hamiltonian.

If $G = (V_0 \sqcup V_1, E)$, then we can transform the EPR Hamiltonian into the **QMC** Hamiltonian by rotating the qubits in V_0 by Y , i.e.

$$(Y \otimes \mathbb{I}) \text{EPR}_{i,j} (Y \otimes \mathbb{I}) = H_{i,j}$$

Bipartite instances and EPR

So about that bipartite guarantee...? First a short detour,

$$\text{EPR} = \sum_{(i,j) \in E} \frac{\mathbb{I} + X_i X_j - Y_i Y_j + Z_i Z_j}{4} = \sum_{(i,j) \in E} |\Phi\rangle\langle\Phi|_{i,j}$$

Why bother with this family of Hamiltonians? Well theoretically interesting but also rotationally (unitarily) related to the original max-cut Hamiltonian.

If $G = (V_0 \sqcup V_1, E)$, then we can transform the EPR Hamiltonian into the **QMC** Hamiltonian by rotating the qubits in V_0 by Y , i.e.

$$(Y \otimes \mathbb{I}) \text{EPR}_{i,j} (Y \otimes \mathbb{I}) = H_{i,j}$$

Turns out that the optimal product state for EPR is much friendlier than **QMC**; with some work one can show that this is just $|0\rangle^{\otimes n}$ or $|1\rangle^{\otimes n}$

Maximum Matching

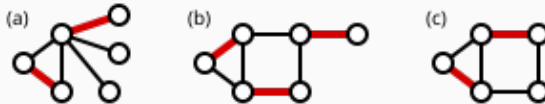


Figure 2: Maximum matchings in three separate graphs

- A **matching** is defined to be a set of edges where no pair shares an adjacent vertex. The maximum matching is just defined as the largest such set in size. (Not to be confused with **maximal matchings**).

Linear program for Maximum Matching

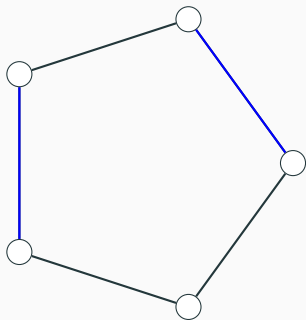
$$\begin{array}{ll}\text{maximize} & \sum_{e \in E} M_e \\ \text{subject to} & \sum_{j \in \Gamma(i)} M_{i,j} \leq 1 \quad \text{for all } i \in V \quad (1)\end{array}$$

$$\sum_{e \in E(S)} M_e \leq \frac{|S|-1}{2} \quad \text{for all } |S| \text{ odd} \quad (2)$$

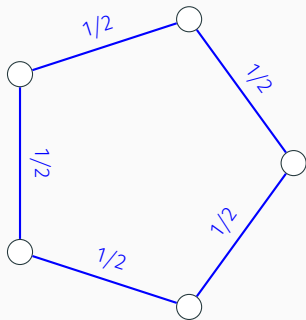
$$M_e \geq 0 \quad \text{for all } e \in E \quad (3)$$

Let's focus on the highlighted inequality. Why is it necessary?

Odd cycles and integrality



(a) max matching for 5-cycle



(b) optimal LP solution

A first attempt

Something that might now seem natural is to do the following.

A first attempt

Something that might now seem natural is to do the following.

- Find a maximum matching $M : E \rightarrow \{0, 1\}$

A first attempt

Something that might now seem natural is to do the following.

- Find a maximum matching $M : E \rightarrow \{0, 1\}$
- For all the edges in the maximum matching, place EPR pairs on those edges. For an unmatched qubit, "do nothing" or place $\mathbb{I}/2$. Define this state to be ρ'

A first attempt

Something that might now seem natural is to do the following.

- Find a maximum matching $M : E \rightarrow \{0, 1\}$
- For all the edges in the maximum matching, place EPR pairs on those edges. For an unmatched qubit, "do nothing" or place $\mathbb{I}/2$. Define this state to be ρ'
- Return $\arg \max_{\{|0^{\otimes n}\rangle\langle 0^{\otimes n}|, \rho'\}} \text{Tr}(\mathbf{H}\rho)$

A first attempt

Something that might now seem natural is to do the following.

- Find a maximum matching $M : E \rightarrow \{0, 1\}$
- For all the edges in the maximum matching, place EPR pairs on those edges. For an unmatched qubit, "do nothing" or place $\mathbb{I}/2$. Define this state to be ρ'
- Return $\arg \max_{\{|0^{\otimes n}\rangle\langle 0^{\otimes n}|, \rho'\}} \text{Tr}(\mathbf{H}\rho)$

Energy of an edge: If $M_e = 1$, and the edge is matched; then the energy of our state ρ with respect to that edge is $\text{Tr}(\mathbf{H}_e\rho) = 1$. What if $M_e = 0$? Then a partial trace argument provides that $\text{Tr}(\mathbf{H}_e\rho) = 1/4$. So putting this all together provides

$$\text{Tr}(\mathbf{H}_e\rho) = \frac{1}{4} + \frac{3M_e}{4}$$

"Fractionality" is scary but is it really?

"Fractionality" is scary but is it really?

That was great! But can we do without the scaling down? What if we eliminated constraint 2 (the one forcing integrality), how much harder does life get?

"Fractionality" is scary but is it really?

That was great! But can we do without the scaling down? What if we eliminated constraint 2 (the one forcing integrality), how much harder does life get?

- Might seem daunting to eliminate constraint forcing integrality since landscape of "fractional solutions" could be massive.

"Fractionality" is scary but is it really?

That was great! But can we do without the scaling down? What if we eliminated constraint 2 (the one forcing integrality), how much harder does life get?

- Might seem daunting to eliminate constraint forcing integrality since landscape of "fractional solutions" could be massive.
- Life is simpler actually. If one denotes the optimal solution to the LP as OPT_f , then

"Fractionality" is scary but is it really?

That was great! But can we do without the scaling down? What if we eliminated constraint 2 (the one forcing integrality), how much harder does life get?

- Might seem daunting to eliminate constraint forcing integrality since landscape of "fractional solutions" could be massive.
- Life is simpler actually. If one denotes the optimal solution to the LP as OPT_f , then

Theorem

For any graph $2OPT_f$ is an integer. Moreover there exists a fractional matching that attains this value, i.e.

$$\sum_{e \in E} f(e) = OPT_f$$

where $f(e) \in \{0, 1/2, 1\}$. (The fractional matching polytope is half-integral).

Prelude to our algorithm

Prelude to our algorithm

- So we have this linear program for maximum matching. it now returns an assignment to edges that is no longer just $\{0, 1\}$ but $\{0, 1/2, 1\}$.

Prelude to our algorithm

- So we have this linear program for maximum matching. it now returns an assignment to edges that is no longer just $\{0, 1\}$ but $\{0, 1/2, 1\}$.
- It turns out one can completely **characterize** what the subgraph of $1/2$ edges look like.

Prelude to our algorithm

- So we have this linear program for maximum matching. it now returns an assignment to edges that is no longer just $\{0, 1\}$ but $\{0, 1/2, 1\}$.
- It turns out one can completely **characterize** what the subgraph of $1/2$ edges look like.
- Just by virtue of the matching constraints, this subgraph has bounded degree (≤ 2). What this means is that it just consists of paths and cycles!

Prelude to our algorithm

- So we have this linear program for maximum matching. it now returns an assignment to edges that is no longer just $\{0, 1\}$ but $\{0, 1/2, 1\}$.
- It turns out one can completely **characterize** what the subgraph of $1/2$ edges look like.
- Just by virtue of the matching constraints, this subgraph has bounded degree (≤ 2). What this means is that it just consists of paths and cycles!
- Moreover the edges that span the cut between this $1/2$ bad-subgraph and the good portion of the starting graph have matching value 0.

The punchline

The punchline

So a natural algorithm to come up with here would be

- Find the maximum matching $M : E \rightarrow \{0, 1/2, 1\}$
- Put EPR pairs on all the edges in the matching
- ??? with edges inscribed by 1/2
- Return $\arg \max_{\{|0^{\otimes n}\rangle\langle 0^{\otimes n}|, \rho'\}} \text{Tr}(H\rho)$

The punchline

So a natural algorithm to come up with here would be

- Find the maximum matching $M : E \rightarrow \{0, 1/2, 1\}$
- Put EPR pairs on all the edges in the matching
- ??? with edges inscribed by 1/2
- Return $\arg \max_{\{|0^{\otimes n}\rangle\langle 0^{\otimes n}|, \rho'\}} \text{Tr}(H\rho)$

From the previous slide, we know that edges spanning the associated cut have matching value 0, so just try optimizing separately.

The punchline

So a natural algorithm to come up with here would be

- Find the maximum matching $M : E \rightarrow \{0, 1/2, 1\}$
- Put EPR pairs on all the edges in the matching
- ??? with edges inscribed by 1/2
- Return $\arg \max_{\{|0^{\otimes n}\rangle\langle 0^{\otimes n}|, \rho'\}} \text{Tr}(H\rho)$

From the previous slide, we know that edges spanning the associated cut have matching value 0, so just try optimizing separately.

Therefore sufficient to find "high-energy" states on paths and cycles that are encompassed in this bad subgraph.

Question: What is the optimal eigenvector of the EPR Hamiltonian for the path graph?

Question: What is the optimal eigenvector of the EPR Hamiltonian for the path graph?

Answer: Turns out a high-energy state for the line is one that is suggested by maximum matching if our input graph was just a line. Obtain a maximum matching on the line, provided by alternating edges of the line and place EPR pairs on them. This state has energy $5n/8$ where n is the number of edges in the path.

Question: What is the optimal eigenvector of the EPR Hamiltonian for the path graph?

Answer: Turns out a high-energy state for the line is one that is suggested by maximum matching if our input graph was just a line. Obtain a maximum matching on the line, provided by alternating edges of the line and place EPR pairs on them. This state has energy $5n/8$ where n is the number of edges in the path.

This question is a little more complicated for the cycle...

Question: What is the optimal eigenvector of the EPR Hamiltonian for the path graph?

Answer: Turns out a high-energy state for the line is one that is suggested by maximum matching if our input graph was just a line. Obtain a maximum matching on the line, provided by alternating edges of the line and place EPR pairs on them. This state has energy $5n/8$ where n is the number of edges in the path.

This question is a little more complicated for the cycle...

In spirit, is similar to the one provided by the high-energy state on the path. Need to take a few extra steps to ensure high energy since we have an extra edge. But can obtain a state with the same energy guarantee as the path.

Proofs (sketches)

Definition

(Level-2 Quantum Lasserre SDP)

$$\text{maximize } \sum_{(i,j) \in E} \frac{1}{4} \cdot \langle v(\mathbb{I}), v(\mathbb{I}) - v(\mathbf{X}_i \mathbf{X}_j) - v(\mathbf{Y}_i \mathbf{Y}_j) - v(\mathbf{Z}_i \mathbf{Z}_j) \rangle$$

$$\text{subject to } v(\mathbf{P}) \in \mathbb{R}^{|\mathcal{P}_2(n)|}$$

$$\forall P \in \mathcal{P}_2(n)$$

$$\langle v(\mathbf{P}), v(\mathbf{P}) \rangle = 1$$

$$\forall P \in \mathcal{P}_2(n)$$

$$\langle v(\mathbf{P}_1), v(\mathbf{Q}_1) \rangle = \langle v(\mathbf{P}_2), v(\mathbf{Q}_2) \rangle$$

$$P_1 Q_1 = P_2 Q_2$$

$$\langle v(\mathbf{P}), v(\mathbf{Q}) \rangle = 0$$

$$PQ = -QP$$

A final note on SDP solution values

A final note on SDP solution values

Let $G = (V, E)$ be a graph. Let $(v(\mathbf{P}))_{\mathbf{P} \in \mathcal{P}_2(n)}$ be a feasible solution for the SDP. Define further

Definition

$$g_{ij} := \frac{1}{4} \cdot \langle v(\mathbb{I}), v(\mathbb{I}) - v(\mathbf{X}_i \mathbf{X}_j) - v(\mathbf{Y}_i \mathbf{Y}_j) - v(\mathbf{Z}_i \mathbf{Z}_j) \rangle$$

$$h_{ij} := g_{ij} - \frac{1}{2}$$

A final note on SDP solution values

Let $G = (V, E)$ be a graph. Let $(v(\mathbf{P}))_{\mathbf{P} \in \mathcal{P}_2(n)}$ be a feasible solution for the SDP. Define further

Definition

$$g_{ij} := \frac{1}{4} \cdot \langle v(\mathbb{I}), v(\mathbb{I}) - v(\mathbf{X}_i \mathbf{X}_j) - v(\mathbf{Y}_i \mathbf{Y}_j) - v(\mathbf{Z}_i \mathbf{Z}_j) \rangle$$
$$h_{ij} := g_{ij} - \frac{1}{2}$$

In particular the objective value of the SDP solution is given by

$$\nu = \sum_{(i,j) \in E} g_{ij}.$$

Monogamy of entanglement

Theorem

Given a feasible solution to the level-2 Lasserre SDP on $G = (V, E)$, then for all $i, j, k \in V$, then

$$h_{ij}^+ + h_{jk}^+ + h_{ik}^+ \leq 1/2$$

Theorem

Given a feasible solution to the level-2 SDP on a graph $G = (V, E)$, then for any vertex $i \in V$ and any $S \subseteq V$,

$$\sum_{j \in S} h_{ij} \leq 1/2$$

and in particular

$$\sum_{j \in \Gamma(i)} h_{ij}^+ \leq 1/2$$

Outline of a working strategy

Outline of a working strategy

Now, something beautiful happens, these solution vectors provided by the SDP can exactly be interpreted as feasible solutions for the linear program. Formally the vector $(2\mathbf{h}_{ij}^+)$ satisfies the matching constraint due to monogamy of entanglement on a star. Non-negativity is built in by construction.

Outline of a working strategy

Now, something beautiful happens, these solution vectors provided by the SDP can exactly be interpreted as feasible solutions for the linear program. Formally the vector $(2\mathbf{h}_{ij}^+)$ satisfies the matching constraint due to monogamy of entanglement on a star.

Non-negativity is built in by construction.

However, we don't quite satisfy (2). Monogamy of entanglement on a triangle tells us that we only satisfy constraint 2 for $|S| = 3$.

Outline of a working strategy

Now, something beautiful happens, these solution vectors provided by the SDP can exactly be interpreted as feasible solutions for the linear program. Formally the vector $(2\mathbf{h}_{ij}^+)$ satisfies the matching constraint due to monogamy of entanglement on a star.

Non-negativity is built in by construction.

However, we don't quite satisfy (2). Monogamy of entanglement on a triangle tells us that we only satisfy constraint 2 for $|S| = 3$.

If we could figure out a way to make the $\mathbf{h}_{ij}^+$'s fully compliant with constraint 2, would provide a feasible solution to the LP.

Scaling down is the answer

Scaling down is the answer

Theorem

If $(\mathbf{x}_e)$ satisfies constraints 1, 3 and 2 for $|S| = 3$, then $(\frac{4}{5}\mathbf{x}_e)$ is feasible for the matching LP.

Scaling down is the answer

Theorem

If $(\mathbf{x}_e)$ satisfies constraints 1, 3 and 2 for $|S| = 3$, then $(\frac{4}{5}\mathbf{x}_e)$ is feasible for the matching LP.

Corollary

The vector $(\frac{8}{5}h_{ij}^+)$ is feasible for the matching LP.

Scaling down is the answer

Theorem

If $(\mathbf{x}_e)$ satisfies constraints 1, 3 and 2 for $|S| = 3$, then $(\frac{4}{5}\mathbf{x}_e)$ is feasible for the matching LP.

Corollary

The vector $(\frac{8}{5}h_{ij}^+)$ is feasible for the matching LP.

So any SDP solution is "almost feasible" for the linear program

Scaling down is the answer

Theorem

If $(\mathbf{x}_e)$ satisfies constraints 1, 3 and 2 for $|S| = 3$, then $(\frac{4}{5}\mathbf{x}_e)$ is feasible for the matching LP.

Corollary

The vector $(\frac{8}{5}h_{ij}^+)$ is feasible for the matching LP.

So any SDP solution is "almost feasible" for the linear program

The crucial insight is that we can now provide a lower bound on the energy of our matching state simply by scaling down the SDP solution and therefore obtain the optimal value to the problem by writing it in terms or as some function of this SDP solution. (Useful for computing the approximation ratio of our algorithm)

Energy of the matching state (attempt 1)

It is now time to compute the energy of the Hamiltonian with respect to the matching state. From the expression for the energy of an edge w.r.t the state, we obtain.

Energy of the matching state (attempt 1)

It is now time to compute the energy of the Hamiltonian with respect to the matching state. From the expression for the energy of an edge w.r.t the state, we obtain.

$$\text{Tr}(\mathbf{H}\rho) = \sum_{e \in E} \left(\frac{1}{4} + \frac{3}{4}M_e \right)$$

Energy of the matching state (attempt 1)

It is now time to compute the energy of the Hamiltonian with respect to the matching state. From the expression for the energy of an edge w.r.t the state, we obtain.

$$\begin{aligned}\mathrm{Tr}(\mathbf{H}\rho) &= \sum_{e \in E} \left(\frac{1}{4} + \frac{3}{4} M_e \right) \\ &\geq \sum_{e \in E} \left(\frac{1}{4} + \frac{6}{5} h_e^+ \right)\end{aligned}$$

Energy of the matching state (attempt 1)

It is now time to compute the energy of the Hamiltonian with respect to the matching state. From the expression for the energy of an edge w.r.t the state, we obtain.

$$\begin{aligned}\mathrm{Tr}(\mathbf{H}\rho) &= \sum_{e \in E} \left(\frac{1}{4} + \frac{3}{4} M_e \right) \\ &\geq \sum_{e \in E} \left(\frac{1}{4} + \frac{6}{5} h_e^+ \right) \\ &\dots\end{aligned}$$

Energy of the matching state (attempt 1)

It is now time to compute the energy of the Hamiltonian with respect to the matching state. From the expression for the energy of an edge w.r.t the state, we obtain.

$$\begin{aligned}\mathrm{Tr}(\mathbf{H}\rho) &= \sum_{e \in E} \left(\frac{1}{4} + \frac{3}{4} M_e \right) \\ &\geq \sum_{e \in E} \left(\frac{1}{4} + \frac{6}{5} h_e^+ \right) \\ &\dots \\ &\dots\end{aligned}$$

Energy of the matching state (attempt 1)

It is now time to compute the energy of the Hamiltonian with respect to the matching state. From the expression for the energy of an edge w.r.t the state, we obtain.

$$\text{Tr}(\mathbf{H}\rho) = \sum_{e \in E} \left(\frac{1}{4} + \frac{3}{4} M_e \right)$$

$$\geq \sum_{e \in E} \left(\frac{1}{4} + \frac{6}{5} h_e^+ \right)$$

...

...

$$\implies \text{Tr}(\mathbf{H}\rho_{\text{final}}) \geq f(\text{SDP}) = \eta \cdot \text{SDP}$$

How do we deal with fractional edges

How do we deal with fractional edges

Before when writing down an expression for the energy, the expression $1/4 + 3M_e/4$, it was crucial that $M_e \in \{0, 1\}$. How do we cope with $M_e = 1/2$.

How do we deal with fractional edges

Before when writing down an expression for the energy, the expression $1/4 + 3M_e/4$, it was crucial that $M_e \in \{0, 1\}$. How do we cope with $M_e = 1/2$.

By our energy estimation, if S is the subgraph of edges with matching value $1/2$, then

$$\sum_{C \subseteq S} \frac{5|C|}{8} = \sum_{C \subseteq S} |C| \left(\frac{1}{4} + \frac{3}{4}M_e \right) = \sum_{e \in S} \left(\frac{1}{4} + \frac{3}{4}M_e \right)$$

How do we deal with fractional edges

Before when writing down an expression for the energy, the expression $1/4 + 3M_e/4$, it was crucial that $M_e \in \{0, 1\}$. How do we cope with $M_e = 1/2$.

By our energy estimation, if S is the subgraph of edges with matching value $1/2$, then

$$\sum_{C \subseteq S} \frac{5|C|}{8} = \sum_{C \subseteq S} |C| \left(\frac{1}{4} + \frac{3}{4}M_e \right) = \sum_{e \in S} \left(\frac{1}{4} + \frac{3}{4}M_e \right)$$

It's now smooth sailing from here. Denoting our solution vectors as $(2h_e^+)_{e \in E}$, since they are feasible for the LP, they must have energy at most the optimal solution. Now

$$\text{Tr}(\mathbf{H}\rho) \geq \sum_{e \in E} \left(\frac{1}{4} + \frac{3}{4}M_e \right) \geq \sum_{e \in E} \left(\frac{1}{4} + \frac{3}{4}(2h_e^+) \right)$$

...

$$\implies \text{Tr}(\mathbf{H}\rho_{\text{final}}) \geq 0.75 \cdot \text{SDP}$$

Some Open Questions

- Can we find a **quantum** approximation algorithm that does better for QMC and EPR respectively? Current SOTA is 0.611 and 0.8395 respectively.
- Goemans-Williamson is optimal upto the Unique Games Conjecture. A work by Hwang, Neeman, Parekh, Thompson and Wright [3] demonstrates hardness of approximation to within a factor of 0.956 assuming a conjecture in Gaussian geometry. Can we refine these conditional hardness results?
- Recent results by [7] demonstrate NP-hardness of approximating QMC for graphs of bounded and constant degree (pretty big deal!)
- Can we demonstrate QMA-hardness of approximation?
- Big outstanding problem on the hardness of the EPR hamiltonian, hasn't shown to be QMA or even NP-hard.

QMA (Quantum Merlin Arthur)

A different perspective on NP

One can think of NP as the class where the following interactive protocol can be setup

Given some string x and a decision problem $\mathcal{L}$, there exists an all-powerful prover that produces a "proof" $\pi \in \{0, 1\}^n$ and sends it to a verifier V (poly-time deterministic algorithm) that must return 1 if x is a YES instance and 0 otherwise.

Definition

QMA has a similar setup where a decision problem $\mathcal{L}$ is said to be in **QMA** iff there exists a quantum algorithm V for which for all possible inputs x , if x is a YES-instance, then there exists a quantum state for which $V(x, |\psi\rangle) = 1$ with probability $\geq 3/4$ and if x is a NO-instance; then for every $|\psi\rangle$, the probability with which V accepts is $\leq 1/4$

We define the decision version of the k -local Hamiltonian problem as follows:

Definition

Assume we have an n -qubit Hamiltonian $\mathbf{H} = \sum_a \mathbf{H}_a$ where each $\mathbf{H}_a$ acts on at most k qubits. Given this Hamiltonian along with two numbers a and b , where $b - a > \frac{1}{\text{poly}(n)}$, decide if the lowest eigenvalue is $\leq a$ or $\geq b$. The former corresponds to the YES instance.

Theorem

k -local Hamiltonian is QMA-complete for all $k \geq 2$.

Encoding problems in NP

In 3-SAT, we are given some formula f on n variables in 3-CNF, i.e. the conjunction of many clauses c_j :

$$f(x_1, \dots, x_n) = c_1 \wedge c_2 \wedge \dots \wedge c_m \quad \text{where } c_j = (x_a \vee \overline{x_b} \vee x_c) \text{ for example}$$

We're now hoping to make a corresponding Hamiltonian $\mathbf{H}_j$ for each clause (where the local terms penalize assignments that don't satisfy a given clause). In the above example, put $\mathbf{H}_j := |010\rangle\langle 010|$ in order to penalize the assignment state $|010\rangle$. Finally define $\mathbf{H} = \sum_{i=1}^m \mathbf{H}_i$.

In particular smallest eigenvalue of $\mathbf{H}$ is exactly the smallest number of unsatisfied clauses. Therefore $\mathbf{H}$ has a 0 eigenvalue $\Leftrightarrow f$ is satisfiable.

Non-Commutative MaxCut

$$\max \sum_{(i,j) \in E} \frac{1 - \text{tr}(X_i X_j)}{2}$$

such that X_i, X_j have eigenvalues ± 1

Main advantage of considering non-commutative CSP's instead of "quantum" CSP's as we've discussed so far is that these are also purely algebraic objects and similar techniques for analyzing to classical CSP's have more promise of working whereas Q-CSP's have the same theory of approximation.

Thanks!

Questions?

A primer on SDPs

Recall that in classical max-cut, the goal was to optimize

$$\sum_{\substack{(i,j) \in E \\ x_i \in \{\pm 1\}}} \frac{1 - x_i x_j}{2}$$

This was NP-hard so can't optimize over this efficiently.

Let's **relax** this instead. Try solving

$$\sum_{\substack{(i,j) \in E \\ \|\mathbf{x}_i\|=1}} \frac{1 - \langle \mathbf{x}_i, \mathbf{x}_j \rangle}{2}$$

Question: Why is this a relaxation?

Answer: Suffices to take $\mathbf{x}_i = (x_i, 0, \dots, 0)$.

- Compute the optimal SDP vectors specified by the $\mathbf{x}_i$'s on the previous slide.
- Remains to round each of these $\mathbf{x}_i$'s to a random sign $\{\pm 1\}$. We do so as follows
 - Pick a random $\mathbf{r} = (r_1, \dots, r_n)$ where $r_i \sim \mathcal{N}(0, 1)$.
 - Finally set the answer to be the vector g where

$$g_i = \text{sign}(\langle \mathbf{r}, \mathbf{x}_i \rangle)$$

Theorem

Goemans-Williamson achieves a 0.878 approximation to Max-Cut.

Back to **Quantum** Max-Cut. How do we write the SDP for QMC. It's actually a bit complicated, but after simplifying, we obtain

$$\text{QMC}_{\text{SDP}}(G) = \sum_{(i,j) \in E} \frac{1 - 3\langle \mathbf{v}_i, \mathbf{v}_j \rangle}{4}$$

Looks very similar to the original original Max-Cut SDP! How do we round?

- Compute solution vectors from the SDP ($\mathbf{v}_i$)
- "Round this" into a Bloch-sphere assignment
 - Initialize a random 3-dimensional projector $\Pi \in \mathbb{R}^{3 \times n}$
 - Set for all $i \in V$,

$$u_i = \Pi \mathbf{v}_i / \|\Pi \mathbf{v}_i\|$$

- Our final state is a product state of ρ_i 's where

$$\rho_i = \frac{1}{2}(\mathbb{I} + u_{i,1}\mathbf{X} + u_{i,2}\mathbf{Y} + u_{i,3}\mathbf{Z})$$



A. Anshu, D. Gosset, and K. Morenz.

Beyond product state approximations for a quantum analogue of max cut.

Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2020.



S. Gharibian and O. Parekh.

Almost optimal classical approximation algorithms for a quantum generalization of max-cut.

Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2019.



Y. Hwang, J. Neeman, O. Parekh, K. Thompson, and J. Wright.

Unique games hardness of quantum max-cut, and a conjectured vector-valued borell's inequality, 2022.



R. King.

An improved approximation algorithm for quantum max-cut on triangle-free graphs.

Quantum, 7:1180, Nov. 2023.



E. Lee and O. Parekh.

An improved quantum max cut approximation via matching, 2024.



O. Parekh and K. Thompson.

An optimal product-state approximation for 2-local quantum hamiltonians with positive terms, 2022.



S. Piddock.

Quantum max-cut is np hard to approximate, 2025.