A matching based algorithm for Quantum Max-Cut

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(Classical) Max-Cut

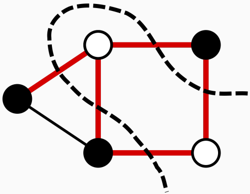

Figure 1: max cut is highlighted in red; size 5

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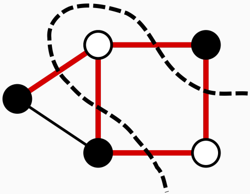

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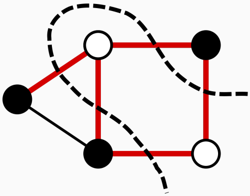

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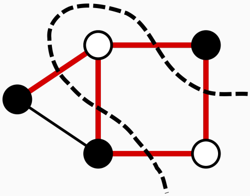

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- Intuitively, terms in the sum only have non-zero contribution when $z_i \neq z_j$, so the z_i's of the same sign identify the partition.
- Known to be **NP-hard** to obtain an exact solution :(

Local Hamiltonians

Local Hamiltonians

- Given some $G = (V, E)$ on n vertices, we can think of the vertices as qubits where the edges of this "interaction graph" describe our local hamiltonian terms, i.e.

$$\mathsf{H} = \sum_{(u,v)\in E} \mathsf{H}_{u,v} \otimes \mathbb{I}_{V\setminus\{u,v\}}$$

Goal: Find a quantum state $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$ that returns the maximum/minimum energy or eigenvalue of H, i.e.

$$\text{energy} = \langle\psi| \mathsf{H} |\psi\rangle$$

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- One of the quantum analogues of classically well-understood constraint satisfaction problems.

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$$\mathsf{SWAP}_{i,j} = \frac{\mathbb{I} + \mathsf{X}_i\mathsf{X}_j + \mathsf{Y}_i\mathsf{Y}_j + \mathsf{Z}_i\mathsf{Z}_j}{2} \implies \mathsf{H}_{i,j} = \frac{\mathbb{I} - \mathsf{SWAP}_{i,j}}{2}$$

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Exactly the anti-symmetric subspace! So energy is maximal when $\mathsf{SWAP}_{i,j} |\psi\rangle = -|\psi\rangle$

Some more quantum intuition

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$$H = \sum_{(i,j)\in E} \frac{\mathbb{I} - X_i X_j - Y_i Y_j - Z_i Z_j}{4}$$

- The $X_i X_j$ terms makes a measurement in the X basis.
    - -1 if both are the same eigenstate, i.e. $|{++}\rangle$ or $|{--}\rangle$.
    - $+1$ if both are in differing eigenstates $|{-+}\rangle, |{+-}\rangle$.

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  - -1 if both are the same eigenstate, i.e. $|{+}{+}\rangle$ or $|{-}{-}\rangle$.
  - $+1$ if both are in differing eigenstates $|{-}{+}\rangle, |{+}{-}\rangle$.
- In order to maximize, The Y_iY_j term should be different in the Y basis and similarly for the Z_iZ_j term.
- Like classical max-cut but not just in the Z basis.

Approximation algorithms for QMC

- Product state algorithm that uses an SDP approach to obtain a 0.498 approximation by Gharibian and Parekh [2].
- Tensor product of one and two qubit states that achieves a 0.531 approximation ratio by Anshu, Gosset and Morenz [1].
- Parekh and Thompson provided a 0.533 approximation algorithm in 2020 [6]
- The first algorithm to incorporate entanglement was due to King in 2022 achieving an approximation ratio of 0.582 for triangle-free graphs. [4]
- Finally Parekh and Lee provide a 0.595 matching-based approximation algorithm [5]

Our idea

Like Parekh and Lee, we look to maximum matching within graphs and manage to improve their analysis to yield a 0.606-approximation algorithm for quantum max-cut, and if the graph is promised to be bipartite, we can guarantee a 0.75 approximation.

Crucially, the result for bipartite instances may be somewhat surprising to folks (at least it was to me) since classically these are the easiest instances of the problem.

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$$\text{EPR} = \sum_{(i,j)\in E} \frac{\mathbb{I} + \mathsf{X}_i\mathsf{X}_j - \mathsf{Y}_i\mathsf{Y}_j + \mathsf{Z}_i\mathsf{Z}_j}{4} = \sum_{(i,j)\in E} |\Phi\rangle\langle\Phi|_{i,j}$$

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If $G = (V_0 \sqcup V_1, E)$, then we can transform the EPR Hamiltonian into the QMC Hamiltonian by rotating the qubits in V_0 by Y, i.e.

$$(\mathsf{Y} \otimes \mathbb{I})\mathrm{EPR}_{i,j}(\mathsf{Y} \otimes \mathbb{I}) = \mathsf{H}_{i,j}$$

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Turns out that the optimal product state for EPR is much friendlier than **QMC**; with some work one can show that this is just $|0\rangle^{\otimes n}$ or $|1\rangle^{\otimes n}$

Maximum Matching

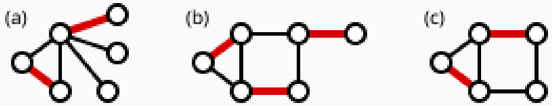


Figure 2: Maximum matchings in three separate graphs

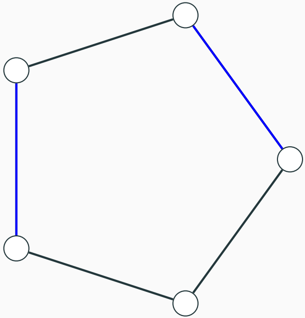
- A **matching** is defined to be a set of edges where no pair shares an adjacent vertex. The maximum matching is just defined as the largest such set in size. (Not to be confused with **maximal matchings**).

Linear program for Maximum Matching

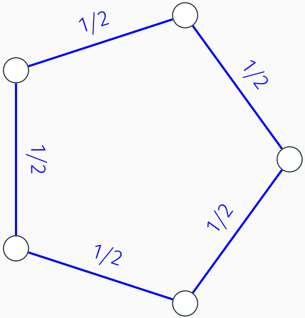
$$
\begin{aligned}
\text{maximize} \quad & \sum_{e \in E} M_e & & \\
\text{subject to} \quad & \sum_{j \in \Gamma(i)} M_{i,j} \leq 1 & \text{for all } i \in V \quad & (1) \\
& \boxed{\sum_{e \in E(S)} M_e \leq \frac{|S|-1}{2}} & \text{for all } |S| \text{ odd} \quad & (2) \\
& M_e \geq 0 & \text{for all } e \in E \quad & (3)
\end{aligned}
$$

Let's focus on the highlighted inequality. Why is it necessary?

Odd cycles and integrality



(a) max matching for 5-cycle



(b) optimal LP solution

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Energy of an edge: If $M_e = 1$, and the edge is matched; then the energy of our state ρ with respect to that edge is $\mathrm{Tr}(\mathsf{H}_e\rho) = 1$. What if $M_e = 0$? Then a partial trace argument provides that $\mathrm{Tr}(\mathsf{H}_e\rho) = 1/4$. So putting this all together provides

$$\mathrm{Tr}(\mathsf{H}_e\rho) = \frac{1}{4} + \frac{3M_e}{4}$$

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Theorem
For any graph $2OPT_f$ is an integer. Moreover there exists a fractional matching that attains this value, i.e.

$$\sum_{e \in E} f(e) = OPT_f$$

where $f(e) \in \{0, 1/2, 1\}$. (The fractional matching polytope is half-integral).

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- It turns out one can completely **characterize** what the subgraph of 1/2 edges look like.
- Just by virtue of the matching constraints, this subgraph has bounded degree (≤ 2). What this means is that it just consists of paths and cycles!
- Moreover the edges that span the cut between this 1/2 bad-subgraph and the good portion of the starting graph have matching value 0.

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- Find the maximum matching $M : E \to \{0, 1/2, 1\}$
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Therefore sufficient to find ”high-energy” states on paths and cycles that are encompassed in this bad subgraph.

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Answer: Turns out a high-energy state for the line is one that is suggested by maximum matching if our input graph was just a line. Obtain a maximum matching on the line, provided by alternating edges of the line and place EPR pairs on them. This state has energy $5n/8$ where n is the number of edges in the path.

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In spirit, is similar to the one provided by the high-energy state on the path. Need to take a few extra steps to ensure high energy since we have an extra edge. But can obtain a state with the same energy guarantee as the path.

Proofs (sketches)

Level-2 SDP

Definition
(Level-2 Quantum Lasserre SDP)

$$
\begin{aligned}
\text{maximize} \quad & \sum_{(i,j)\in E} \frac{1}{4} \cdot \langle v(\mathbb{I}), v(\mathbb{I}) - v(\mathsf{X}_i\mathsf{X}_j) - v(\mathsf{Y}_i\mathsf{Y}_j) - v(\mathsf{Z}_i\mathsf{Z}_j)\rangle & \\
\text{subject to} \quad & v(\mathsf{P}) \in \mathbb{R}^{|\mathcal{P}_2(n)|} & \forall P \in \mathcal{P}_2(n) \\
& \langle v(\mathsf{P}), v(\mathsf{P})\rangle = 1 & \forall P \in \mathcal{P}_2(n) \\
& \langle v(\mathsf{P}_1), v(\mathsf{Q}_1)\rangle = \langle v(\mathsf{P}_2), v(\mathsf{Q}_2)\rangle & P_1Q_1 = P_2Q_2 \\
& \langle v(\mathsf{P}), v(\mathsf{Q})\rangle = 0 & PQ = -QP
\end{aligned}
$$

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Let $G = (V, E)$ be a graph. Let $(v(\mathsf{P}))_{\mathsf{P} \in \mathcal{P}_2(n)}$ be a feasible solution for the SDP. Define further

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In particular the objective value of the SDP solution is given by $\nu = \sum_{(i,j)\in E} g_{ij}$.

Monogamy of entanglement

Theorem
Given a feasible solution to the level-2 Lasserre SDP on $G = (V, E)$, then for all $i, j, k \in V$, then

$$h^+_{ij} + h^+_{jk} + h^+_{ik} \leq 1/2$$

Theorem
Given a feasible solution to the level-2 SDP on a graph $G = (V, E)$, then for any vertex $i \in V$ and any $S \subseteq V$,

$$\sum_{j \in S} h_{ij} \leq 1/2$$

and in particular

$$\sum_{j \in \Gamma(i)} h^+_{ij} \leq 1/2$$

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Now, something beautiful happens, these solution vectors provided by the SDP can exactly be interpreted as feasible solutions for the linear program. Formally the vector $(2\mathsf{h}_{ij}^{+})$ satisfes the matching constraint due to monogamy of entagelement on a star.
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However, we don't quite satisfy (2). Monogamy of entanglement on a triangle tells us that we only satisfy constraint 2 for $|S| = 3$.

If we could figure out a way to make the h_{ij}^{+}'s fully compliant with constraint 2, would provide a feasible solution to the LP.

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The crucial insight is that we can now provide a lower bound on the energy of our matching state simply by scaling down the SDP solution and therefore obtain the optimal value to the problem by writing it in terms or as some function of this SDP solution. (Useful for computing the approximation ratio of our algorithm)

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$$\begin{aligned} \mathrm{Tr}(\mathsf{H}\rho) &= \sum_{e\in E}\left(\frac{1}{4}+\frac{3}{4}M_e\right) \\ &\geq \sum_{e\in E}\left(\frac{1}{4}+\frac{6}{5}h_e^+\right) \\ &\dots \\ &\dots \end{aligned}$$

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$$\begin{aligned} \mathsf{Tr}(\mathsf{H}\rho) &= \sum_{e \in E} \left(\frac{1}{4} + \frac{3}{4} M_e \right) \\ &\geq \sum_{e \in E} \left(\frac{1}{4} + \frac{6}{5} h_e^+ \right) \\ &\cdots \\ &\cdots \\ \implies \mathsf{Tr}(\mathsf{H}\rho_{\text{final}}) &\geq f(\mathsf{SDP}) = \eta \cdot \mathsf{SDP} \end{aligned}$$

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By our energy estimation, if S is the subgraph of edges with matching value 1/2, then

$$\sum_{C \subseteq S} \frac{5|C|}{8} = \sum_{C \subseteq S} |C| \left(\frac{1}{4} + \frac{3}{4} M_e \right) = \sum_{e \in S} \left(\frac{1}{4} + \frac{3}{4} M_e \right)$$

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It's now smooth sailing from here. Denoting our solution vectors as $(2h_e^+)_{e \in E}$, since they are feasible for the LP, they must have energy at most the optimal solution. Now

$$\mathrm{Tr}(\mathrm{H}\rho) \geq \sum_{e \in E} \left(\frac{1}{4} + \frac{3}{4} M_e \right) \geq \sum_{e \in E} \left(\frac{1}{4} + \frac{3}{4} (2\mathbf{h}_e^+) \right)$$

$$\ldots$$

$$\implies \mathrm{Tr}(\mathrm{H}\rho_{\text{final}}) \geq 0.75 \cdot \mathrm{SDP}$$

Some Open Questions

- Can we find a **quantum** approximation algorithm that does better for QMC and EPR respectively?
- Goemans-Williamson is optimal upto the Unique Games Conjecture. A work by Hwang, Neeman, Parekh, Thompson and Wright [3] demonstrates hardness of approximation to within a factor of 0.956 assuming a conjecture in Gaussian geometry. Can we refine these conditional hardness results?
- Can we demonstrate QMA-hardness of approximation?
- Big outstanding problem on the hardness of the EPR hamiltonian, hasn't shown to be QMA or even NP-hard.

Thanks!

Questions?

A primer on SDPs

Goemans-Williamson

Recall that in classical max-cut, the goal was to optimize

$$\sum_{\substack{(i,j)\in E \\ x_i\in\{\pm 1\}}} \frac{1 - x_i x_j}{2}$$

This was NP-hard so can't optimize over this efficiently.

Let's **relax** this instead. Try solving

$$\sum_{\substack{(i,j)\in E \\ \|\mathbf{x}_i\|=1}} \frac{1 - \langle \mathbf{x}_i, \mathbf{x}_j \rangle}{2}$$

Question: Why is this a relaxation?

Answer: Suffices to take $\mathbf{x}_i = (x_i, 0, \ldots, 0)$.

Goemans-Williamson

- Compute the optimal SDP vectors specified by the $\mathbf{x}_i$'s on the previous slide.
- Remains to round each of these $\mathbf{x}_i$'s to a random sign $\{\pm 1\}$. We do so as follows
    - Pick a random $\mathbf{r} = (r_1, \ldots, r_n)$ where $r_i \sim \mathcal{N}(0, 1)$.
    - Finally set the answer to be the vector g where

$$g_i = \mathrm{sign}(\langle \mathbf{r}, \mathbf{x}_i \rangle)$$

Theorem
Goemans-Williamson achieves a 0.878 *approximation to Max-Cut.*

Gharibian-Parekh

Back to **Quantum** Max-Cut. How do we write the SDP for QMC. It's actually a bit complicated, but after simplifying, we obtain

$$\mathrm{QMC}_{\mathsf{SDP}}(G) = \sum_{(i,j)\in E} \frac{1 - 3\langle \mathbf{v}_i, \mathbf{v}_j \rangle}{4}$$

Looks very similar to the original original Max-Cut SDP! How do we round?

- Compute solution vectors from the SDP (v_i)
- "Round this" into a Bloch-sphere assignment
  - Initialize a random 3-dimensional projector $\Pi \in \mathbb{R}^{3\times n}$
  - Set for all $i \in V$,

$$u_i = \Pi v_i / \|\Pi v_i\|$$

- Our final state is a product state of ρ_i's where

$$\rho_i = \frac{1}{2}(\mathbb{I} + u_{i,1}\mathsf{X} + u_{i,2}\mathsf{Y} + u_{i,3}\mathsf{Z})$$

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