Error estimates and asymptotic analysis for exact qudit universality

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Overview

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Preliminaries and some background

Question

Given that we have a quantum state $|\psi\rangle \in \mathbb{C}^d$ and a unitary matrix $\mathbf{U} \in \mathbb{C}^{d \times d}$, can living in a larger Hilbert space can help us describe a significantly larger set of unitary evolutions?

</> **Definition**

A qu(d)it is a quantum version of d-ary digits where the state can be described by a vector in a d-dimensional $\mathcal{H}_d$ Hilbert space. The basis vectors are denoted by $|0\rangle, |1\rangle, \cdots, |d-1\rangle$, and the state of a qudit has the general form

$$|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle \in \mathbb{C}_d \text{ and } \sum_{i=0}^{d-1} |\alpha_i|^2 = 1$$

An example when $d = 1$ is the uniform superposition of 1 and 0, i.e.

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

Preliminaries and some background

</> Definition

A Givens rotation is represented by a matrix of the form

$$\mathbf{G}(i,j,\theta) = \begin{bmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & c & \dots & -s & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & s & \dots & c & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & s & \dots & c & \dots & 0 \end{bmatrix}$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$. Here $c, s \in \mathbb{R}$ but generalizations to $c, s \in \mathbb{C}$ can be made. Intuitively, a Givens rotation is a rotation between the i-th and j-th axes in the overarching d-dimensional Hilbert space $\mathcal{H}^d$.

The actual problem of universality

Theorem

$\mathbf{G}(i, j, \theta)$ is orthonormal and its generalizations with complex valued entries are unitary, i.e.

$$\mathbf{G}^\dagger \mathbf{G} = \mathbf{G}\mathbf{G}^\dagger = \mathbf{I}$$

Goal

Given any $\mathbf{U} \in \mathbf{C}^{d \times d}$, can we write $\mathbf{U}$ down as a product of rotations acting on a 2-dimensional subspace of $\mathcal{H}^d$, i.e.

$$\mathbf{U} = \mathbf{U}_{i_1, j_1} \mathbf{U}_{i_2, j_2} \cdots \mathbf{U}_{i_k, j_k}$$

Since "orthonormality" is preserved under taking products, a corollary of this is that the product of d Givens rotations is also unitary.

As it turns out yes, we can! There exists something known as the **QR decomposition** where a matrix can be decomposed into a productx of orthogonal and upper triangular matrices and it turns out the Givens rotations are the building blocks of this decomposition.

A sketch of universality

Fact

Any unitary $\mathbf{U} \in \mathbf{C}^{d \times d}$ can be written as

$$\mathbf{U} = \mathbf{QR}$$

where $\mathbf{Q}$ is a product of d Givens rotations and $\mathbf{R}$ is a diagonal matrix

Proof: The intuition here is that pre-multiplying on the left of $\mathbf{U}$ by an appropriate Givens rotation that for some (i, j) zeroes out those specific entry. Note we only need to consider the matrix entries of $\mathbf{U}$ corresponding to the i-th and j-th columns (since the rest of the columns are unaffected by the rotation). Indeed, we let $\mathbf{U}_{i,j} \in \mathbb{R}^{2 \times 2}$ be the the block-matrix that gets affected by the Givens rotation.

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} u_{i,i} & u_{i,j} \\ u_{j,i} & u_{j,j} \end{bmatrix} = \begin{bmatrix} r_{i,i} & r_{i,j} \\ 0 & r_{j,j} \end{bmatrix}$$

An end to universality

The above matrix equality gives us a set of four equations namely

$$ca_{i,i} + sa_{j,i} = r_{i,i} \quad \text{and} \quad ca_{i,j} + sa_{j,j} = r_{i,j}$$
$$-sa_{i,i} + ca_{j,i} = 0 \quad \text{and} \quad -sa_{i,j} + ca_{j,j} = r_{j,j}$$

Solving the above set of equations and noting that $c^2 + s^2 = 1$ in order to preserve orthonormality of $\mathbf{Q}$, we obtain some familiar looking formulas for c, s namely

$$c = \frac{a_{i,i}}{\sqrt{a_{i,i}^2 + a_{j,i}^2}} \quad \text{and} \quad s = \frac{a_{j,i}}{\sqrt{a_{i,i}^2 + a_{j,i}^2}}$$

Time-complexity

For any $\mathbf{U} \in \mathbf{C}^{d \times d}$, we may get unlucky and have to single out every single entry below the main diagonal; which requires

$$\sum_{i=1}^{d-1} i = \frac{d(d-1)}{2} = \binom{d}{2} \in \mathcal{O}(d^2)$$

Connectivity Graphs and toy architectures

</> Definition

For a qu(d)it, we can define a connectivity graph where each node represents a level of the qudit and the edges represent the possible transitions between the levels.

Algorithm

1. `input`: $\mathbf{U} \in \mathbb{C}^{d\times d}, \mathcal{G} = (V, E)$
2. `decompose`: Generate the set of required givens rotations

$$\mathcal{R} = \{\mathbf{G}_{i,j} \in \mathbb{C}^{d\times d} \mid 2 \leq i < j \leq d-1\}$$

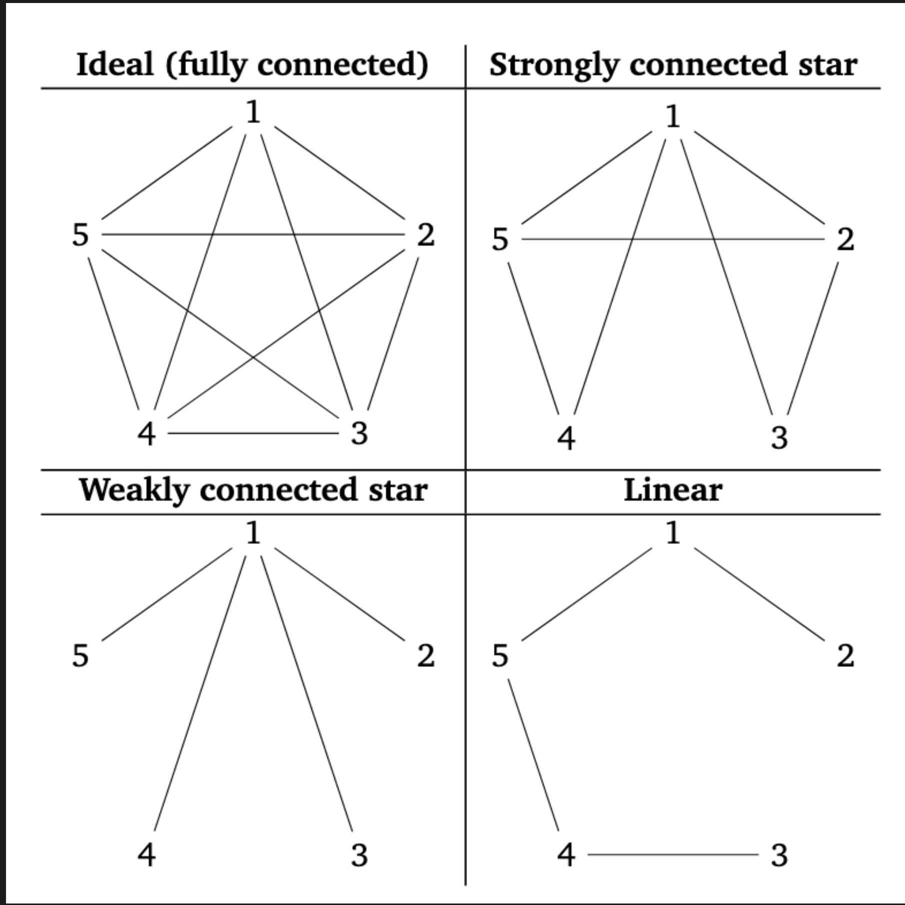
such that $\mathbf{G}_{2,1} \ldots \mathbf{G}_{d,d-1}\mathbf{U} = \mathbf{R}$

3. `augmentation` : For each $\mathbf{G}_{i,j}$
   - `if` $(i, j) \in E$, `do` nothing
   - `else` find shortest path between i and j $(i, v_1, \ldots, v_{k-1}, v_k, j)$

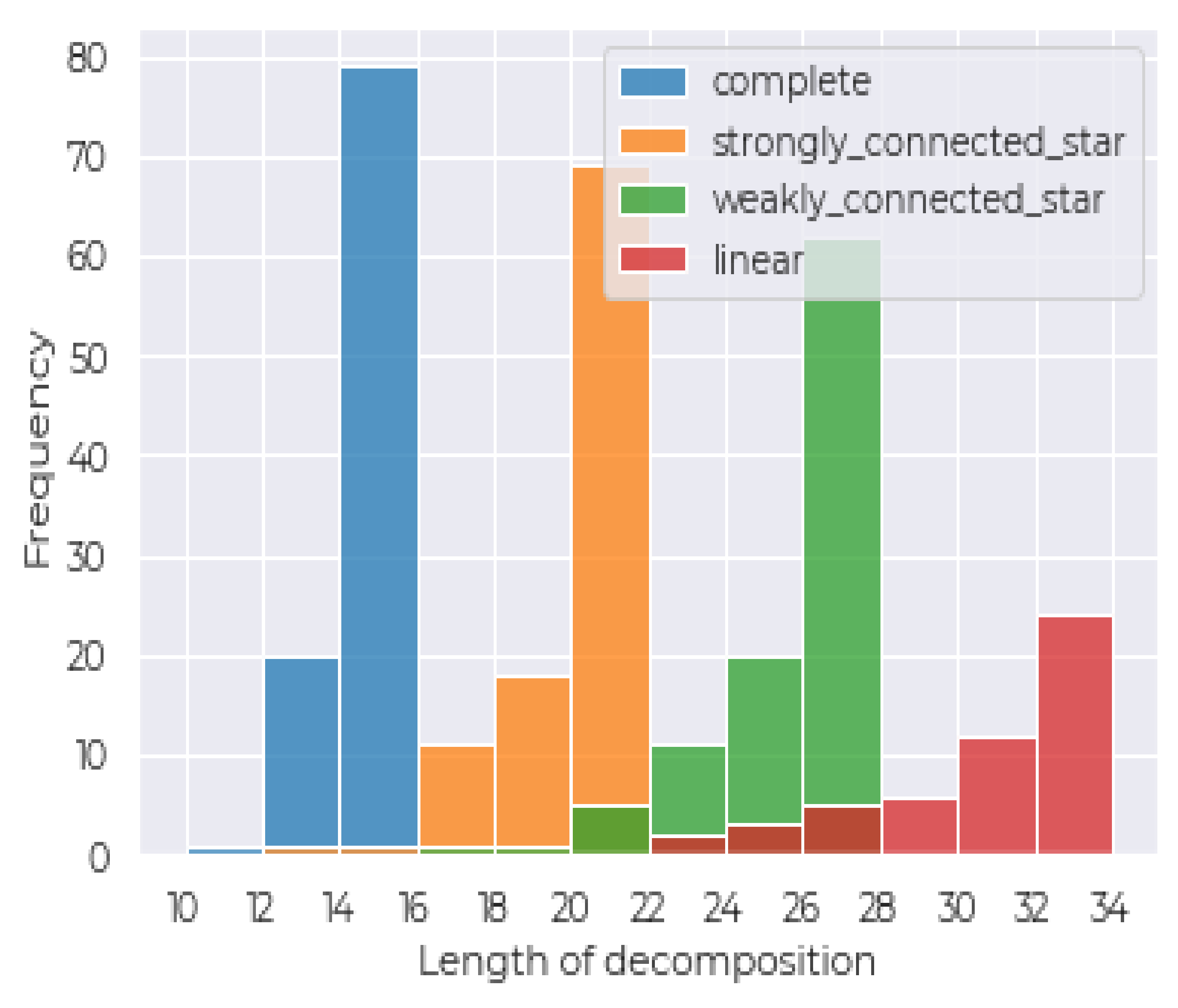
$$\mathbf{G}_{i,j} = \mathcal{S}_{i,v_1} \ldots \mathcal{S}_{v_{k-1},v_k} \mathbf{G}_{v_k,j} \mathcal{S}_{v_{k-1},v_k} \ldots \mathcal{S}_{i,v_1}$$

4. `return`: $\mathcal{S}_{\ell,m}, \mathbf{G}_{v_k,j}, \mathbf{R}$

Following some motivation via the $^{40}\mathrm{Ca}^{+}$ ion trap and some other trapped ion designs, we consider the following toy architectures:



Graph decomposition lengths



An experimental error-model

Each givens rotation can be written as $\exp(i\theta\mathbf{H})$ for some hermitian $\mathbf{H}$ and some angle θ. Since we've dealt with decomposition length we can now focus on primaryily fidelity. Assume that physically we can realize each givens rotation with some rotation angle error namely $\exp(i(\theta+\delta)\mathbf{H})$. More concretely let k be the decomposition length of the unitary that we'd like to implement, thereby giving us

$$\mathbf{U}_{\text{ideal}} = \prod_{i=1}^{k} \exp(i\theta_i\mathbf{H}_\mathbf{i}) = \prod_{i=1}^{k} \mathbf{U}_i$$

$$\mathbf{U}_{\text{ion}} = \prod_{i=1}^{k} \exp(i(\theta_i+\delta_i)\mathbf{H}_i) = \prod_{i=1}^{k} \exp(i\theta_i\mathbf{H}_i)\exp(i\delta_i\mathbf{H}_i) = \prod_{i=1}^{k} \mathbf{U}_i\mathcal{E}_i$$

An experimental error-model

Given some arbitary starting state $|\psi\rangle \in \mathbb{C}^d$, we have that

$$\begin{aligned}
\mathcal{F} &= \left| \langle\psi| \mathbf{U}_{\text{ideal}}^{\dagger} \mathbf{U}_{\text{ion}} |\psi\rangle \right| = \left| \langle\psi| \left(\prod_{j=1}^{k} \mathbf{U}_j \right)^{\dagger} \left(\prod_{j=1}^{k} \mathbf{U}_j \mathcal{E}_j \right) |\psi\rangle \right| \\
&\geq \left| \langle\psi| \prod_{j=1}^{k} \mathcal{E}_j |\psi\rangle \right| \geq \left| \langle\psi| \exp\left(i \cdot \sum_{j=1}^{k} \delta_j \cdot \mathbf{H}_j \right) |\psi\rangle \right| \\
&= \left| \langle\psi| \exp\left(i \cdot k\delta_{\text{avg}} \cdot \mathcal{H}_{\text{avg}} \right) |\psi\rangle \right| \\
&= \left| \langle\psi| \cos(k\delta_{\text{avg}})\mathbf{I} + i\sin(k\delta_{\text{avg}})\mathcal{H}_{\text{avg}} |\psi\rangle \right| \approx \left(1 - \frac{1}{2} k^2 \delta_{\text{avg}}^2 \right)
\end{aligned}$$

Errors in a quantum computation build linearly

</> Definition

Given two quantum states ρ, σ in terms of their density matrices, their trace distance is defined as

$$||\rho - \sigma||_{\mathrm{tr}} = \frac{1}{2} \sup_{\mathbf{U} \in \mathbb{C}^{n \times n}} \mathrm{trace} \left| \mathbf{U} \rho \mathbf{U}^{\dagger} - \mathbf{U} \sigma \mathbf{U}^{\dagger} \right|$$

where the supremum is over all $n \times n$ unitary matrices and the absolute value of a matrix is defined as the matrix with the same entries as the original matrix but with all entries replaced by their absolute values.

Let $\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_T$ be a sequence of unitary operators; each representing the ideal unitary operator at time t. Let $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_T$ be the noisy approximations to $\mathbf{U_i}$ that we actually implement.

Errors in a quantum computation build linearly

Theorem

Suppose $||\mathbf{U}_i \rho \mathbf{U}_i^\dagger - \mathbf{V}_i \rho \mathbf{V}_i^\dagger||_{\text{tr}} \leq \varepsilon$ for all i and mixed states ρ. Then

$$||\mathbf{U}_T \ldots \mathbf{U}_1 \rho \mathbf{U}_1^\dagger \ldots \mathbf{U}_T^\dagger - \mathbf{V}_T \ldots \mathbf{V}_1 \rho \mathbf{V}_1^\dagger \ldots \mathbf{V}_T^\dagger||_{\text{tr}} \leq \varepsilon T$$

Proof

We argue for the case $T = 2$.

$$\begin{aligned}
||\mathbf{U}_2\mathbf{U}_1\rho\mathbf{U}_1^\dagger\mathbf{U}_2^\dagger - \mathbf{V}_2\mathbf{V}_1\rho\mathbf{V}_1^\dagger\mathbf{V}_2^\dagger||_{\text{tr}} &= ||\mathbf{U}_2\mathbf{U}_1\rho\mathbf{U}_1^\dagger\mathbf{U}_2^\dagger - \mathbf{V}_2\mathbf{U}_1\rho\mathbf{U}_1^\dagger\mathbf{V}_2^\dagger + \mathbf{V}_2\mathbf{U}_1\rho\mathbf{U}_1^\dagger\mathbf{V}_2^\dagger - \mathbf{V}_2\mathbf{V}_1\rho\mathbf{V}_1^\dagger\mathbf{V}_2^\dagger||_{\text{tr}} \\
&\leq ||\mathbf{U}_2\underbrace{\mathbf{U}_1\rho\mathbf{U}_1^\dagger}_{\rho'}\mathbf{U}_2^\dagger - \mathbf{V}_2\underbrace{\mathbf{U}_1\rho\mathbf{U}_1^\dagger}_{\rho'}\mathbf{V}_2^\dagger||_{\text{tr}} + ||\mathbf{V}_2\mathbf{U}_1\rho\mathbf{U}_1^\dagger\mathbf{V}_2^\dagger - \mathbf{V}_2\mathbf{V}_1\rho\mathbf{V}_1^\dagger\mathbf{V}_2^\dagger||_{\text{tr}} \\
&\leq ||\mathbf{U}_2\rho'\mathbf{U}_2^\dagger - \mathbf{V}_2\rho'\mathbf{V}_2^\dagger||_{\text{tr}} + ||\mathbf{V}_2(\mathbf{U}_1\rho\mathbf{U}_1^\dagger - \mathbf{V}_1\rho\mathbf{V}_1^\dagger)\mathbf{V}_2^\dagger||_{\text{tr}} \\
&= ||\mathbf{U}_2\rho'\mathbf{U}_2^\dagger - \mathbf{V}_2\rho'\mathbf{V}_2^\dagger||_{\text{tr}} + ||\mathbf{U}_1\rho\mathbf{U}_1^\dagger - \mathbf{V}_1\rho\mathbf{V}_1^\dagger||_{\text{tr}} \\
&\leq 2\varepsilon
\end{aligned}$$

The general case easily follows by induction.

Thank you and acknowledgements

- **Personal links**:
  - link to github
  - reach out!
  - website

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Extras

Details that were swept under the rug

Baker-Campbell-Hausdorff formula

Given two matrices $\mathbf{A}, \mathbf{B}$, the Baker-Campbell-Hausdorff formula gives us a value for C that solves the equation; $\exp(\mathbf{A})\exp(\mathbf{B}) = \exp(\mathbf{C})$ for possible non-commuting $\mathbf{A}, \mathbf{B}$. The formula is given by

$$\mathbf{C} = \mathbf{A} + \mathbf{B} + \frac{1}{2}[\mathbf{A}, \mathbf{B}] + \frac{1}{12}[\mathbf{A}, [\mathbf{A}, \mathbf{B}]] - \frac{1}{12}[\mathbf{B}, [\mathbf{A}, \mathbf{B}]] + \cdots$$

For two givens rotations $\mathbf{G}(i, j), \mathbf{G}(k, l)$, the commutator is given by

- $i = k, j = l$: $[\mathbf{G}(i, j), \mathbf{G}(k, l)] = 0$
- $i \neq k, j \neq l$: $[\mathbf{G}(i, j), \mathbf{G}(k, l)] = 0$
- $j = k$: $[\mathbf{G}(i, j), \mathbf{G}(j, l)] = \mathbf{G}'(i, l)$